

MIRROR EXTENSIONS OF RATIONAL VERTEX OPERATOR ALGEBRAS

XINGJUN LIN

ABSTRACT. In this paper, mirror extensions of rational vertex operator algebras are considered. The mirror extension conjecture is proved.

1. INTRODUCTION

Mirror extensions were first studied in the context of conformal nets, new conformal nets were obtained and it was further proved that these conformal nets can not be obtained by cosets, orbifolds and simple current extensions [X]. The mirror extensions of vertex operator algebras were studied in [DJX1], mirror extensions of specific vertex operator algebras were obtained and it was conjectured that the mirror extensions exist for general vertex operator algebras satisfying some conditions. More precisely, let $(U, Y, \mathbf{1}, \omega)$ be a vertex operator algebra and $(V, Y, \mathbf{1}, \omega')$ be a vertex operator subalgebra of U such that $\omega' \in U_2$ and $\omega_2 \omega' = 0$, denote $(V^c, Y, \mathbf{1}, \omega - \omega')$ the commutant vertex operator algebra of V in U (see [FZ]). Assume further that $(V, Y, \mathbf{1}, \omega)$, $(V^c, Y, \mathbf{1}, \omega - \omega')$, $(U, Y, \mathbf{1}, \omega')$ are rational and C_2 -cofinite vertex operator algebras such that $(V^c)^c = V$ and U has the following decomposition as $V \otimes V^c$ -module:

$$U = V \otimes V^c \oplus (\oplus_{i=1}^n M^i \otimes N^i),$$

where M^i 's and N^i 's are irreducible modules for V and V^c respectively. Let

$$V^e = V \oplus (\oplus_{i=1}^n m_i M^i)$$

and

$$(V^c)^e = V^c \oplus (\oplus_{i=1}^n m_i (N^i)'),$$

where m_i 's are nonnegative integers. It was conjectured in [DJX1] that if there is a vertex operator algebra structure on V^e such that V^e is an extension vertex operator algebra of V , then $(V^c)^e$ has a vertex operator algebra structure such that $(V^c)^e$ is an extension vertex operator algebra of V^c , the so-called mirror extension vertex operator algebra of V^c . The main goal of this paper is to prove this conjecture.

In our proof of the mirror extension conjecture, we in fact prove that if $(V, Y, \mathbf{1}, \omega)$, $(V^c, Y, \mathbf{1}, \omega - \omega')$, $(U, Y, \mathbf{1}, \omega')$ are rational and C_2 -cofinite vertex operator algebras such that $(V^c)^c = V$, then U has the following decomposition as $V \otimes V^c$ -module:

$$U = V \otimes V^c \oplus (\oplus_{i=1}^n M^i \otimes N^i) \tag{1.1}$$

where M^i 's and N^i 's are inequivalent irreducible modules for V and V^c respectively. This fact plays an important role in our proof of the mirror extension conjecture. Explicitly, let \mathcal{C}_V , \mathcal{C}_{V^c} be the module categories of V , V^c , respectively. We prove that $\{V, M^i | 1 \leq i \leq n\}$ (resp. $\{V^c, N^i | 1 \leq i \leq n\}$) are closed under the tensor product of \mathcal{C}_V (resp. \mathcal{C}_{V^c}). In particular, $\{V, M^i | 1 \leq i \leq n\}$ (resp. $\{V^c, N^i | 1 \leq i \leq n\}$) forms a braided monoidal subcategory \mathcal{C}_V^0 (resp. $\mathcal{C}_{V^c}^0$) of \mathcal{C}_V (resp. \mathcal{C}_{V^c}). We further prove that the decomposition (1.1) induces a braid-reversing equivalence between \mathcal{C}_V^0 and $\mathcal{C}_{V^c}^0$ such that M^i maps to $(N^i)'$ for any $1 \leq i \leq n$. Then the mirror extension conjecture could be proved immediately by using results in [HKL]. These generalize the results in [OS], where the level-rank duality [F] was studied via tensor category and the category of integrable highest weight representations of affine lie algebras \widehat{sl}_m at level n was proved to be partly braid-reversing equivalent to the category of integrable highest weight representations of affine lie algebras \widehat{sl}_n at level m . From this point of view, V is dual to V^c .

This paper is organized as follows: In Section 2, we recall some basic facts about tensor categories and vertex operator algebras. In Section 3, we prove our main results on mirror extensions of vertex operator algebras.

2. PRELIMINARIES

2.1. Preliminaries on tensor categories. In this subsection, we recall some notions and facts in the theory of tensor categories from [BK], [DMNO], [KO] and [O].

2.1.1. Braided tensor categories. Let \mathcal{C} be a monoidal category defined as in [BK] and denote the unit object in \mathcal{C} by $\mathbf{1}_{\mathcal{C}}$. For an object U in \mathcal{C} , a (*right*) *dual* of U is an object U^* with two morphisms

$$e_U : U^* \otimes U \longrightarrow \mathbf{1}_{\mathcal{C}},$$

$$i_U : \mathbf{1}_{\mathcal{C}} \longrightarrow U \otimes U^*,$$

such that the composition

$$U \xrightarrow{i_U \otimes \text{id}_U} U \otimes U^* \otimes U \xrightarrow{\text{id}_U \otimes e_U} U$$

is equal to id_U and the composition

$$U^* \xrightarrow{\text{id}_{U^*} \otimes i_U} U^* \otimes U \otimes U^* \xrightarrow{e_U \otimes \text{id}_{U^*}} U^*$$

is equal to id_{U^*} . Similarly, for any object $U \in \mathcal{C}$ we could define the *left dual* *U of U . If the dual of U exist, then the dual of U is unique (see [BK]). Furthermore, we have the following results which will be used frequently in this paper (see [BK]).

Proposition 2.1. *Let \mathcal{C} be a monoidal category and V be an object in \mathcal{C} . If V has a dual V^* , then*

$$\mathrm{Hom}(U \otimes V, W) = \mathrm{Hom}(U, W \otimes V^*),$$

$$\mathrm{Hom}(U, V \otimes W) = \mathrm{Hom}(V^* \otimes U, W).$$

Recall [BK] that a monoidal category \mathcal{C} is called a *braided tensor category* if there is a natural bifunctor isomorphism $c_{X,Y} : X \otimes Y \rightarrow Y \otimes X$ called a *braiding* subject to the hexagon axioms. For every braided tensor category (\mathcal{C}, c) , there is a *reversed braiding* on \mathcal{C} given by $c_{X,Y}^{rev} = c_{Y,X}^{-1}$, a braided tensor category \mathcal{C} endowed with the reversed braiding will be denoted \mathcal{C}^{rev} . Two objects X, Y in a braided tensor category \mathcal{C} are said to *mutually centralize* each other, in the sense of [M1], if $c_{X,Y}c_{Y,X} = \mathrm{id}_{X \otimes Y}$. For a subcategory \mathcal{D} of a braided tensor category \mathcal{C} , its *centralizer* \mathcal{D}' is the full subcategory of \mathcal{C} consisting of all objects that centralize every object of \mathcal{D} .

For a monoidal category \mathcal{C} , the *Drinfeld center* $\mathcal{Z}(\mathcal{C})$ is defined to be the category whose objects are pairs $(V, \gamma_{-,V})$ for V an object of \mathcal{C} , and $\gamma_{-,V}$ is a natural family of isomorphisms $\gamma_{X,V} : X \otimes V \rightarrow V \otimes X$ defined for all objects $X \in \mathcal{C}$ such that for all objects $X, Y \in \mathcal{C}$,

$$\gamma_{X \otimes Y, V} = (\gamma_{X,V} \otimes \mathrm{id}_Y)(\mathrm{id}_X \otimes \gamma_{Y,V}).$$

A morphism from $(V, \gamma_{-,V})$ to $(W, \gamma_{-,W})$ is a morphism $f : V \rightarrow W$ in \mathcal{C} such that for every object $X \in \mathcal{C}$,

$$(f \otimes \mathrm{id}_X)\gamma_{X,V} = \gamma_{X,W}(\mathrm{id}_X \otimes f).$$

The following theorem was proved in [Ka].

Theorem 2.2. *For a monoidal category \mathcal{C} , $\mathcal{Z}(\mathcal{C})$ is a braided tensor category and the braiding is given by*

$$\gamma_{V,W} : (V, \gamma_{-,V}) \rightarrow (W, \gamma_{-,W}).$$

Let \mathcal{C}, \mathcal{D} be monoidal categories. If \mathcal{C}, \mathcal{D} are braided tensor categories, then a monoidal functor $\mathcal{T} : \mathcal{C} \rightarrow \mathcal{D}$ is called *braided* if \mathcal{T} preserves the braiding; A braided monoidal functor $\mathcal{T} : \mathcal{C} \rightarrow \mathcal{D}$ is said to be a *braided equivalence of categories* if it is an equivalence of the underlying categories (see [BK]). A *braid-reversing equivalence* $\mathcal{T} : \mathcal{C} \rightarrow \mathcal{D}$ is a braided equivalence $\mathcal{C} \rightarrow \mathcal{D}^{rev}$. Given a monoidal functor $\mathcal{T} : \mathcal{C} \rightarrow \mathcal{D}$ with \mathcal{C} braided, we say \mathcal{T} is *central* if it factors through the forgetful functor $\mathcal{Z}(\mathcal{D}) \rightarrow \mathcal{D}$ via a braided monoidal functor $\mathcal{C} \rightarrow \mathcal{Z}(\mathcal{D})$.

If \mathcal{C} is a braided tensor category, we have the functors

$$\begin{aligned} i : \mathcal{C} &\rightarrow \mathcal{Z}(\mathcal{C}), & X &\mapsto (X, c_{-,X}), \\ j : \mathcal{C} &\rightarrow \mathcal{Z}(\mathcal{C}), & X &\mapsto (X, c_{X,-}^{-1}) \end{aligned}$$

are faithfully braided and braid-reversing monoidal functors, respectively (see [Ka], [M2]).

Recall [BK] that a monoidal category \mathcal{C} is called *rigid* if every object in \mathcal{C} has right and left duals. A monoidal category \mathcal{C} is called a *fusion category* if \mathcal{C} is a semisimple rigid monoidal category with finite dimensional spaces of morphisms, finitely many irreducible objects and an irreducible unit object. The following result was obtained in [M2].

Theorem 2.3. *Let \mathcal{C} be a braided fusion category, i, j be the functors defined above. Then we have*

$$i(\mathcal{C})' = j(\mathcal{C}).$$

2.1.2. Algebras in monoidal categories. Let \mathcal{C} be a monoidal category and denote the unit object in \mathcal{C} by $\mathbf{1}_{\mathcal{C}}$. An *algebra* in \mathcal{C} is an object $A \in \mathcal{C}$ along with morphisms $\mu : A \otimes A \rightarrow A$ and $\iota_A : \mathbf{1}_{\mathcal{C}} \hookrightarrow A$ such that the following conditions hold:

- (i) *Associativity.* Compositions $\mu \circ (\mu \otimes \text{id}) \circ a$, $\mu \circ (\text{id} \otimes \mu) : A \otimes (A \otimes A) \rightarrow A$ are equal, where a denotes the associativity isomorphism $a : A \otimes (A \otimes A) \rightarrow (A \otimes A) \otimes A$;
- (ii) *Unit.* Composition $\mu \circ (\iota_A \otimes A) : A = \mathbf{1}_{\mathcal{C}} \otimes A \rightarrow A$ is equal to id_A ;

This completes the definition of the algebra. We will denote the algebra just defined by (A, μ, ι_A) or briefly by A .

For an algebra A in \mathcal{C} , the *left module category* ${}_A\mathcal{C}$ is defined to be the category whose objects are pairs (M, μ_M) , where $M \in \mathcal{C}$ and $\mu_M : A \otimes M \rightarrow M$ is a morphism in \mathcal{C} such that:

$$\begin{aligned} \mu_M \circ (\mu \otimes \text{id}) \circ a &= \mu_M \circ (\text{id} \otimes \mu_M) : A \otimes (A \otimes M) \rightarrow M; \\ \mu_M(\iota_A \otimes \text{id}) &= \text{id} : \mathbf{1} \otimes M \rightarrow M. \end{aligned}$$

The morphisms are defined by

$$\begin{aligned} &\text{Hom}_{{}_A\mathcal{C}}((M_1, \mu_{M_1}), (M_2, \mu_{M_2})) \\ &= \{\phi \in \text{Hom}_{\mathcal{C}}(M_1, M_2) \mid \mu_{M_2} \circ (\text{id} \otimes \phi) = \phi \circ \mu_{M_1} : A \otimes M_1 \rightarrow M_2\}. \end{aligned}$$

Similarly, we could define the *right module category* \mathcal{C}_A . An algebra A in \mathcal{C} is called *semisimple* if the category ${}_A\mathcal{C}$ is semisimple. The following theorem was proved in [DMNO].

Proposition 2.4. *Let \mathcal{C} be a fusion category and A be an algebra in \mathcal{C} . Then the following conditions are equivalent:*

- (i) *The category \mathcal{C}_A is semisimple;*
- (ii) *The category ${}_A\mathcal{C}$ is semisimple.*

We now assume that \mathcal{C} is a modular tensor category defined as in [BK]. In particular, \mathcal{C} is a braided fusion category. An *etale algebra* A in \mathcal{C} is defined to be an algebra such that the category \mathcal{C}_A is semisimple and A is *commutative*, that is, $\mu \circ c_{A,A} : A \otimes A \rightarrow A$ is equal to μ . In particular, from Proposition 2.4 we have A is semisimple, i.e., the category ${}_A\mathcal{C}$ is semisimple. An etale algebra A is called *connected* if the unit object

appears in A with multiplicity 1. For a connected etale algebra $A \in \mathcal{C}$, it was proved that the category \mathcal{C}_A is a fusion category with operation \otimes_A (see [DMNO]). Let \mathcal{F}_A be the *free module functor* $\mathcal{F}_A : \mathcal{C} \rightarrow \mathcal{C}_A$ defined by $\mathcal{F}_A(X) = X \otimes A$ (see [KO]). It was proved in [KO] that \mathcal{F}_A is a monoidal functor. Furthermore, it was proved that \mathcal{F}_A has the structure of a central functor given by $X \mapsto (X \otimes A, c_{-,X})$ (see [DMNO]).

Recall from [BK] that for any modular tensor category \mathcal{C} and U in \mathcal{C} there is a functorial isomorphism

$$\theta_U : U \longrightarrow U$$

such that:

$$\theta_{U \otimes W} = c_{W,U} c_{U,W} (\theta_U \otimes \theta_W);$$

$$\theta_1 = \text{id};$$

$$\theta_{U^*} = (\theta_U)^*,$$

where $(\theta_U)^* \in \text{Hom}(U^*, U^*)$ denotes the image of $\theta_U \in \text{Hom}(U, U)$ under the canonical map. The following theorem was essentially proved in [KO].

Theorem 2.5. *Let \mathcal{C} be a modular tensor category and A be a commutative algebra in \mathcal{C} such that $\theta_A = \text{id}$. If A is simple as an A -module, then A is semisimple.*

In the proof of our main results, we will frequently use the following result (see [KO], [O]).

Theorem 2.6. *Let \mathcal{C} be a modular tensor category and A be an algebra in \mathcal{C} . For any object $U \in \mathcal{C}$ and left A -module M , we have*

$$\text{Hom}_A(A \otimes U, M) = \text{Hom}(U, M),$$

where we denote $\text{Hom}_{A\mathcal{C}}$ by Hom_A . Similarly, for any object $U \in \mathcal{C}$ and A -module M , we have

$$\text{Hom}_{\mathcal{C}_A}(U \otimes A, M) = \text{Hom}(U, M).$$

2.1.3. Frobenius-Perron dimension. Recall from [ENO] that for a fusion category \mathcal{C} there is a unique homomorphism from the Grothendieck ring of \mathcal{C} to real numbers sending each isomorphism class to a nonnegative real number. The value of this homomorphism on the class represented by $X \in \mathcal{C}$ is called the *Frobenius-Perron dimension* of X and written $\text{FPdim}_{\mathcal{C}}(X)$. For a fusion category \mathcal{C} , one defines its *Frobenius-Perron dimension* as follows:

$$\text{FPdim } \mathcal{C} = \sum_{X \in \mathcal{O}(\mathcal{C})} \text{FPdim}_{\mathcal{C}}(X)^2,$$

where $\mathcal{O}(\mathcal{C})$ is the set of isomorphism classes of simple objects in \mathcal{C} . The following result was obtained in [DMNO].

Proposition 2.7. *Let \mathcal{C} a modular tensor category and A be a connected etale algebra in \mathcal{C} . Then*

$$(\text{FPdim}_{\mathcal{C}} A) \text{FPdim } \mathcal{C}_A = \text{FPdim } \mathcal{C}.$$

2.2. Preliminaries on vertex operator algebras.

2.2.1. Vertex operator algebras. We first recall some basic notions and facts in the theory of vertex operator algebras from [DLM], [FHL], [FLM], [LL] and [Z]. A *vertex operator algebra* is a quadruple $(V, Y, \mathbf{1}, \omega)$, where $V = \bigoplus_{n \in \mathbb{Z}} V_n$ is a \mathbb{Z} -graded \mathbb{C} -vector space such that $\dim V_n < \infty$ and $V_n = 0$ for n sufficiently large, $\mathbf{1} \in V_0$ is the *vacuum vector* of V , ω is the *conformal vector* of V , and Y is a linear map

$$Y : V \rightarrow (\text{End} V)[[x, x^{-1}]],$$

$$v \mapsto Y(v, x) = \sum_{n \in \mathbb{Z}} v_n x^{-n-1}, \quad v_n \in \text{End} V$$

satisfying the following axioms:

- (i) For any $u, v \in V$, $u_n v = 0$ for n sufficiently large;
- (ii) $Y(\mathbf{1}, x) = \text{id}_V$;
- (iii) $Y(v, x)\mathbf{1} \in V[[x]]$ and $Y(v, x)\mathbf{1}|_{x=0} = v$ for any $v \in V$;
- (iv) The component operators of $Y(\omega, x) = \sum_{n \in \mathbb{Z}} L(n)x^{-n-2}$ satisfy the Virasoro algebra relation with *central charge* $c \in \mathbb{C}$:

$$[L(m), L(n)] = (m - n)L(m + n) + \frac{1}{12}(m^3 - m)\delta_{m+n,0}c,$$

and

$$L(0)|_{V_n} = n, \quad n \in \mathbb{Z},$$

$$\frac{d}{dx}Y(v, x) = Y(L(-1)v, x);$$

- (v) The *Jacobi identity*

$$\begin{aligned} & x_0^{-1} \delta \left(\frac{x_1 - x_2}{x_0} \right) Y(u, x_1) Y(v, x_2) - x_0^{-1} \delta \left(\frac{x_2 - x_1}{-x_0} \right) Y(v, x_2) Y(u, x_1) \\ &= x_2^{-1} \delta \left(\frac{x_1 - x_0}{x_2} \right) Y(Y(u, x_0)v, x_2). \end{aligned}$$

This completes the definition of the vertex operator algebra and we will denote the vertex operator algebra briefly by V . A vertex operator algebra V is called of *CFT* type if $V = \bigoplus_{n \geq 0} V_n$ and $\dim V_0 = 1$.

Let V be a vertex operator algebra. A *weak V -module* M is a vector space equipped with a linear map

$$Y_M : V \rightarrow (\text{End} M)[[x, x^{-1}]],$$

$$v \mapsto Y_M(v, x) = \sum_{n \in \mathbb{Z}} v_n x^{-n-1}, \quad v_n \in \text{End} M$$

satisfying the following conditions: For any $u \in V$, $v \in V$, $w \in M$ and $n \in \mathbb{Z}$,

$$u_n w = 0 \text{ for } n \gg 0;$$

$$Y_M(\mathbf{1}, x) = \text{id}_M;$$

$$\begin{aligned} & x_0^{-1} \delta \left(\frac{x_1 - x_2}{x_0} \right) Y_M(u, x_1) Y_M(v, x_2) - x_0^{-1} \delta \left(\frac{x_2 - x_1}{-x_0} \right) Y_M(v, x_2) Y_M(u, x_1) \\ &= x_2^{-1} \delta \left(\frac{x_1 - x_0}{x_2} \right) Y_M(Y(u, x_0)v, x_2). \end{aligned}$$

A weak V -module M is called an *admissible V -module* if M has a $\mathbb{Z}_{\geq 0}$ -gradation $M = \bigoplus_{n \in \mathbb{Z}_{\geq 0}} M(n)$ such that

$$a_m M(n) \subset M(wa + n - m - 1)$$

for any homogeneous $a \in V$ and $m, n \in \mathbb{Z}$. An admissible V -module M is said to be *irreducible* if M has no non-trivial admissible weak V -submodule. When an admissible V -module M is a direct sum of irreducible admissible submodules, M is called *completely reducible*. A vertex operator algebra V is said to be *rational* if any admissible V -module is completely reducible.

A V -module is a weak V -module M which carries a \mathbb{C} -grading induced by the spectrum of $L(0)$, that is $M = \bigoplus_{\lambda \in \mathbb{C}} M_\lambda$ where $M_\lambda = \{w \in M | L(0)w = \lambda w\}$. Moreover one requires that M_λ is finite dimensional and for fixed $\lambda \in \mathbb{C}$, $M_{\lambda+n} = 0$ for sufficiently small integer n . Similarly, a V -module M is said to be *irreducible* if M has no non-trivial V -submodule and a vertex operator algebra V is said to be *regular* if any weak V -module M is a direct sum of irreducible V -modules.

For a V -module $M = \bigoplus_{\lambda \in \mathbb{C}} M(\lambda)$, set $M' = \bigoplus_{\lambda \in \mathbb{C}} M_\lambda^*$, the restricted dual of M . It was proved in [FHL] that M' is naturally a V -module where the vertex operator denoted by Y' is defined by the property

$$\langle Y'(v, x)w', w \rangle = \langle w', Y(e^{xL(1)}(-x^{-2})^{L(0)}v, x^{-1})w \rangle$$

for $v \in V, w' \in M'$ and $w \in M$. The V -module M' is called the *contragredient module* of M . A V -module M is called *self dual* if M is isomorphic to M' .

Recall that a vertex operator algebra V is called *C_2 -cofinite* if $\dim V/C_2(V) < \infty$, where $C_2(V) = \langle u_{-2}v | u, v \in V \rangle$. The following results were obtained in [ABD], [L].

Theorem 2.8. *Let V be a CFT type vertex operator algebra. Then V is regular if and only if V is rational and C_2 -cofinite.*

We now assume that V is a vertex operator algebra satisfying the following conditions:

- (1) V is simple CFT type vertex operator algebra and is self dual;
- (2) V is regular.

Let M be a weak V -module, we have $L(0)$ acts semisimply on M , i.e.

$$M = \bigoplus_{\lambda \in \mathbb{C}} M_\lambda,$$

where $M_\lambda = \{w \in M \mid L(0)w = \lambda w\}$. If $w \in M_\lambda$, we write $\text{wt} w = \lambda$ and call w a *homogeneous vector*. In particular, if M is an irreducible V -module, it was proved in [Z] that

$$M = \bigoplus_{n=0}^{\infty} M_{\lambda+n},$$

for some λ such that $M_\lambda \neq 0$, and λ is defined to be the *conformal weight* of M .

2.2.2. Tensor product vertex operator algebra. Let $(V^1, Y, \mathbf{1}, \omega^1), \dots, (V^p, Y, \mathbf{1}, \omega^p)$ be vertex operator algebras, the tensor product of $(V^1, Y, \mathbf{1}, \omega^1), \dots, (V^p, Y, \mathbf{1}, \omega^p)$ is constructed on the tensor product vector space

$$V = V^1 \otimes \cdots \otimes V^p,$$

where the vertex operator Y is defined by

$$Y(v^1 \otimes \cdots \otimes v^p, x) = Y(v^1, x) \otimes \cdots \otimes Y(v^p, x)$$

for $v^i \in V^i$ ($1 \leq i \leq p$), the vacuum vector is

$$\mathbf{1} = \mathbf{1} \otimes \cdots \otimes \mathbf{1}$$

and the conformal vector is

$$\omega = \omega^1 \otimes \cdots \otimes \mathbf{1} + \cdots + \mathbf{1} \otimes \cdots \otimes \omega^p.$$

Then $(V, Y, \mathbf{1}, \omega)$ is a vertex operator algebra (see [FHL], [LL]).

Let M^i be an admissible V^i -module for $i = 1, \dots, p$. We may construct the tensor product admissible module $M^1 \otimes \cdots \otimes M^p$ for the tensor product vertex operator algebra $V^1 \otimes \cdots \otimes V^p$ by

$$Y_{M^1 \otimes \cdots \otimes M^p}(v^1 \otimes \cdots \otimes v^p, x) = Y_{M^1}(v^1, x) \otimes \cdots \otimes Y_{M^p}(v^p, x).$$

Then $(M^1 \otimes \cdots \otimes M^p, Y_{M^1 \otimes \cdots \otimes M^p})$ is an admissible $V^1 \otimes \cdots \otimes V^p$ -module. Furthermore, we have the following results which were proved in [FHL], [DMZ] and [ABD].

Theorem 2.9. *Let V^1, \dots, V^p be vertex operator algebras. If V^1, \dots, V^p are rational, then $V^1 \otimes \cdots \otimes V^p$ is rational and any irreducible $V^1 \otimes \cdots \otimes V^p$ -module is a tensor product $M^1 \otimes \cdots \otimes M^p$, where M^1, \dots, M^p are some irreducible modules for the vertex operator algebras V^1, \dots, V^p , respectively. If V^1, \dots, V^p are C_2 -cofinite, then $V^1 \otimes \cdots \otimes V^p$ is C_2 -cofinite. In particular, if V^1, \dots, V^p are vertex operator algebras satisfying conditions (1) and (2), then $V^1 \otimes \cdots \otimes V^p$ satisfies conditions (1) and (2).*

2.2.3. *Tensor category of V -modules.* First we recall the notions of intertwining operator and fusion rule from [FHL]. Let V be a vertex operator algebra, M^1, M^2, M^3 be weak V -modules. An *intertwining operator* \mathcal{Y} of type $\begin{pmatrix} M^3 \\ M^1 \ M^2 \end{pmatrix}$ is a linear map

$$\mathcal{Y} : M^1 \rightarrow \text{Hom}(M^2, M^3)\{x\},$$

$$w^1 \mapsto \mathcal{Y}(w^1, x) = \sum_{n \in \mathbb{C}} w_n^1 x^{-n-1}$$

satisfying the following conditions: For any $v \in V, w^1 \in M^1, w^2 \in M^2$ and $\lambda \in \mathbb{C}$,

$$w_{n+\lambda}^1 w^2 = 0 \text{ for } n \gg 0;$$

$$\frac{d}{dx} \mathcal{Y}(w^1, x) = \mathcal{Y}(L(-1)w^1, x);$$

$$x_0^{-1} \delta\left(\frac{x_1 - x_2}{x_0}\right) Y_{M^3}(v, x_1) \mathcal{Y}(w^1, x_2) - x_0^{-1} \delta\left(\frac{x_2 - x_1}{-x_0}\right) \mathcal{Y}(w^1, x_2) Y_{M^2}(v, x_1)$$

$$= x_2^{-1} \delta\left(\frac{x_1 - x_0}{x_2}\right) \mathcal{Y}(Y_{M^1}(v, x_0)w^1, x_2).$$

Denote the vector space of intertwining operators of type $\begin{pmatrix} M^3 \\ M^1 \ M^2 \end{pmatrix}$ by $\mathcal{V}_{M^1, M^2}^{M^3}$.

The dimension of $\mathcal{V}_{M^1, M^2}^{M^3}$ is called the *fusion rule* for M^1, M^2 and M^3 , and is denoted by $N_{M^1, M^2}^{M^3}$. If V is a vertex operator algebra satisfying conditions (1) and (2), M^1, M^2, M^3 are V -modules, we have $N_{M^1, M^2}^{M^3} = N_{(M^1)', (M^2)'}^{(M^3)'}$ (see [DJX2]).

For any intertwining operator \mathcal{Y} of type $\begin{pmatrix} M^3 \\ M^1 \ M^2 \end{pmatrix}$, define the operator \mathcal{Y}^* as follows:

$$\mathcal{Y}^*(w^2, x)w^1 = e^{xL(-1)}\mathcal{Y}(w^1, -x)w^2$$

for any $w^1 \in M^1, w^2 \in M^2$. It was proved in [FHL] that \mathcal{Y}^* is an intertwining operators of type $\begin{pmatrix} M^3 \\ M^2 \ M^1 \end{pmatrix}$. The following theorem was proved in [ADL].

Theorem 2.10. *Let V^1, V^2 be rational vertex operator algebras. Let M^1, M^2, M^3 be V^1 -modules and N^1, N^2, N^3 be V^2 -modules such that*

$$\dim I_{V^1} \begin{pmatrix} M^3 \\ M^1 \ M^2 \end{pmatrix} < \infty, \quad \dim I_{V^2} \begin{pmatrix} N^3 \\ N^1 \ N^2 \end{pmatrix} < \infty.$$

Then the linear map

$$\sigma : I_{V^1} \begin{pmatrix} M^3 \\ M^1 \ M^2 \end{pmatrix} \otimes I_{V^2} \begin{pmatrix} N^3 \\ N^1 \ N^2 \end{pmatrix} \rightarrow I_{V^1 \otimes V^2} \begin{pmatrix} M^3 \otimes N^3 \\ M^1 \otimes N^1 \ M^2 \otimes N^2 \end{pmatrix}$$

$$\mathcal{Y}_1 \otimes \mathcal{Y}_2 \mapsto (\mathcal{Y}_1 \otimes \mathcal{Y}_2)(\cdot, x)$$

is an isomorphism, where $(\mathcal{Y}_1 \otimes \mathcal{Y}_2)(\cdot, x)$ is defined by

$$(\mathcal{Y}_1 \otimes \mathcal{Y}_2)(\cdot, x)(u^1 \otimes v^1, x)u^2 \otimes v^2 = \mathcal{Y}_1(u^1, x)u^2 \otimes \mathcal{Y}_2(v^1, x)v^2.$$

We now turn our discussions to the tensor category of V -modules. Let M^1, M^2 be weak V -modules. A *tensor product* for the ordered pair (M^1, M^2) is a pair (M, \mathcal{Y}^{12}) consisting of a weak V -module M and an intertwining operator \mathcal{Y}^{12} of type $\begin{pmatrix} M \\ M^1 \ M^2 \end{pmatrix}$ satisfying the following universal property: For any weak V -module W and any intertwining operator I of type $\begin{pmatrix} W \\ M^1 \ M^2 \end{pmatrix}$, there exists a unique V -homomorphism ψ from M to W such that $I = \psi \circ \mathcal{Y}^{12}$.

Fix a complex number $z \in \mathbb{C}^\times$, for any \mathbb{C} -graded V -module $M = \oplus_{n \in \mathbb{C}} M_n$ such that $\dim M_n < \infty$ for each $n \in \mathbb{C}$, set

$$\overline{M} = \prod_{n \in \mathbb{C}} M_n.$$

For any V -modules M^1, M^2, M^3 , a $P(z)$ -intertwining map F of type $\begin{pmatrix} M^3 \\ M^1 \ M^2 \end{pmatrix}$ is a linear map

$$F : M^1 \otimes M^2 \rightarrow \overline{M^3}$$

such that for any $v \in V, w^1 \in M^1, w^2 \in M^2$,

$$\begin{aligned} & z^{-1} \delta\left(\frac{x_1 - x_0}{z}\right) F(Y_{M^1}(v, x_0) w^1 \otimes w^2) + x_0^{-1} \delta\left(\frac{z - x_1}{-x_0}\right) F(w^1 \otimes Y_{M^2}(v, x_1) w^2) \\ &= x_0^{-1} \delta\left(\frac{x_1 - z}{x_0}\right) Y_{M^3}(v, x_1) F(w^1 \otimes w^2). \end{aligned}$$

Let M^1 and M^2 be V -modules. A $P(z)$ -tensor product of M^1 and M^2 is a pair

$$(M^1 \boxtimes_{P(z)} M^2, \boxtimes_{P(z)})$$

consisting of a V -module $M^1 \boxtimes_{P(z)} M^2$ and a $P(z)$ -intertwining map $\boxtimes_{P(z)}$ of type $\begin{pmatrix} M^1 \boxtimes_{P(z)} M^2 \\ M^1 \ M^2 \end{pmatrix}$ satisfying the following universal property: For any pair (M^3, F) consisting of a V -module M^3 and a $P(z)$ -intertwining map F of type $\begin{pmatrix} M^3 \\ M^1 \ M^2 \end{pmatrix}$, there is a unique V -module morphism η from $M^1 \boxtimes_{P(z)} M^2$ to M^3 such that $G = \bar{\eta} \circ F$, where $\bar{\eta}$ is the map from $\overline{M^3}$ to $\overline{M^4}$ uniquely extending η . The V -module $M^1 \boxtimes_{P(z)} M^2$ is called a $P(z)$ -tensor product module of M^1 and M^2 .

We now recall from [HL4] the precise connection between intertwining operators and $P(z)$ -intertwining maps of the same type. Fix an integer p . For any V -modules M^1, M^2, M^3 , let \mathcal{Y} be an intertwining operator of type $\begin{pmatrix} M^3 \\ M^1 \ M^2 \end{pmatrix}$. We have a linear map $F_{\mathcal{Y}, p} : M^1 \otimes M^2 \rightarrow \overline{M^3}$ given by

$$F_{\mathcal{Y}, p}(w^1 \otimes w^2) = \mathcal{Y}(w^1, e^{l_p(z)} w^2)$$

for any $w^1 \in M^1, w^2 \in M^2$, where $l_p(z) = \log z + 2p\pi i$. Conversely, given a $P(z)$ -intertwining map F of type $\begin{pmatrix} M^3 \\ M^1 \ M^2 \end{pmatrix}$, for any homogeneous elements $w^1 \in M^1$, $w^2 \in M^2$ and complex number $n \in \mathbb{C}$, let $w_n^1 w^2$ be the projection of the image of $w^1 \otimes w^2$ under F to the homogeneous subspace of M^3 of weight $\text{wt} w^1 + \text{wt} w^2 - n - 1$, multiplied by $e^{(n+1)l_p(z)}$. Set

$$\mathcal{Y}_{F,p}(w^1, x)w^2 = \sum_{n \in \mathbb{C}} w_n^1 w^2 x^{-n-1},$$

we obtain a linear map

$$\begin{aligned} M^1 \otimes M^2 &\rightarrow M^3\{x\} \\ w^1 \otimes w^2 &\mapsto \mathcal{Y}_{F,p}(w^1, x)w^2. \end{aligned}$$

The following theorem was established in [HL3], [HL4].

Theorem 2.11. *For $p \in \mathbb{Z}$, the correspondence $\mathcal{Y} \mapsto F_{\mathcal{Y},p}$ is a linear isomorphism from the vector space of intertwining operators of type $\begin{pmatrix} M^3 \\ M^1 \ M^2 \end{pmatrix}$ to the vector space of $P(z)$ -intertwining maps of type $\begin{pmatrix} M^3 \\ M^1 \ M^2 \end{pmatrix}$. Its inverse map is given by $F \mapsto \mathcal{Y}_{F,p}$.*

In the following, we take $p = 0$ and the meaning of the intertwining operator associated to a $P(z)$ -intertwining map is clear.

In general, the existence of a $P(z)$ -tensor product is not obvious. If V is a vertex operator algebra satisfying conditions (1) and (2), and let M^1, \dots, M^k be the complete list of irreducible V -modules. It was proved in a series of papers [HL1], [HL2], [HL3] [H1] and [H3] that the $P(z)$ -tensor product $(M^1 \boxtimes_{P(z)} M^2, \boxtimes_{P(z)})$ exists for any two V -modules M^1, M^2 and $M^1 \boxtimes_{P(z)} M^2 \cong \oplus_i N_{M^1, M^2}^{M^i} M^i$. From Theorem 2.11, there is a unique intertwining operator $\mathcal{Y}_{P(z); M^1, M^2}$ of type $\begin{pmatrix} M^1 \boxtimes_{P(z)} M^2 \\ M^1 \ M^2 \end{pmatrix}$ associated to the $P(z)$ -intertwining map $\boxtimes_{P(z)}$. From the definition of $P(z)$ -tensor product, it is clear that $(M^1 \boxtimes_{P(z)} M^2, \mathcal{Y}_{P(z); M^1, M^2})$ is a tensor product of M^1 and M^2 . In the following, we denote the $P(1)$ -tensor product $(M^1 \boxtimes_{P(1)} M^2, \boxtimes_{P(1)})$ by $(M^1 \boxtimes M^2, \boxtimes_{P(1)})$ and $\mathcal{Y}_{P(1); M^1, M^2}$ by \mathcal{Y}_{M^1, M^2} . For any $w^1 \in M^1$ and $w^2 \in M^2$, the $P(z)$ -tensor product of w^1 and w^2 is defined to be

$$w^1 \boxtimes_{P(z)} w^2 = \mathcal{Y}_{P(z); M^1, M^2}(w^1, x)w^2|_{x^n = e^{n \log z}}, n \in \mathbb{C}.$$

We now recall some facts about the braiding isomorphism from [H4]. Let $z_1, z_2 \in \mathbb{C}^\times$ and γ be a path in \mathbb{C}^\times from z_1 to z_2 . The *parallel isomorphism* $\mathcal{T}_\gamma : M^1 \boxtimes_{P(z_1)} M^2 \rightarrow M^1 \boxtimes_{P(z_2)} M^2$ (see [HL4], [H4]) is defined as follows: Let $\mathcal{Y}_{P(z_2); M^1, M^2}$ be the intertwining operator associated to the $P(z_2)$ -tensor product $(M^1 \boxtimes_{P(z_2)} M^2, \boxtimes_{P(z_2)})$ and $l(z_1)$ be the

value of logarithm of z_1 determined uniquely by $\log z_2$ and the path γ . Then \mathcal{T}_γ is characterized by

$$\overline{\mathcal{T}_\gamma}(w^1 \boxtimes_{P(z_1)} w^2) = \mathcal{Y}_{P(z_2); M^1, M^2}(w^1, x) w^2|_{x^n = e^{n\log(z_1)}}, n \in \mathbb{C}$$

for any $w^1 \in M^1$ and $w^2 \in M^2$. The parallel isomorphism depends only on the homotopy class of γ .

For any V -modules M^1, M^2 , the *braiding isomorphism* (also called the *commutativity isomorphism*)

$$C_{P(z); M^1, M^2} : M^1 \boxtimes_{P(z)} M^2 \rightarrow M^2 \boxtimes_{P(z)} M^1$$

is characterized as follows: For any $w^1 \in M^1$ and $w^2 \in M^2$,

$$\overline{C_{P(z); M^1, M^2}}(w^1 \boxtimes_{P(z)} w^2) = e^{zL(-1)} \overline{\mathcal{T}_{\gamma_z^-}}(w^2 \boxtimes_{P(-z)} w^1),$$

where γ_z^- is a path from $-z$ to z in the closed upper half plane with 0 deleted and $\mathcal{T}_{\gamma_z^-}$ is the corresponding parallel isomorphism. When $z = 1$, denote the braiding isomorphism $C_{P(1); M^1, M^2} : M^1 \boxtimes M^2 \rightarrow M^2 \boxtimes M^1$ by C_{M^1, M^2} .

Let $\mathcal{Y}_{M^1, M^2}, \mathcal{Y}_{M^2, M^1}$ be the intertwining operators associated to the $P(1)$ -tensor product $(M^1 \boxtimes M^2, \boxtimes_{P(1)})$, $(M^2 \boxtimes M^1, \boxtimes_{P(1)})$, respectively. Then $(M^1 \boxtimes M^2, \mathcal{Y}_{M^1, M^2})$ is a tensor product of M^1 and M^2 , $(M^2 \boxtimes M^1, \mathcal{Y}_{M^2, M^1})$ is a tensor product of M^2 and M^1 . Consider the intertwining operator \mathcal{Y}_{M^2, M^1}^* of type $\begin{pmatrix} M^2 \boxtimes M^1 \\ M^1 \ M^2 \end{pmatrix}$ which is defined by

$$\mathcal{Y}_{M^2, M^1}^*(w^1, x) w^2 = e^{xL(-1)} \mathcal{Y}_{M^2, M^1}(w^2, -x) w^1.$$

Lemma 2.12. *For any V -modules M^1, M^2 and $w^1 \in M^1, w^2 \in M^2$, we have*

$$\overline{C_{M^1, M^2}}(\mathcal{Y}_{M^1, M^2}(w^1, x) w^2|_{x^n = e^{n \log 1}}) = \mathcal{Y}_{M^2, M^1}^*(w^1, -x) w^2|_{x^n = e^{n \pi i}}.$$

Proof: By direct calculation, for any $w^1 \in M^1$ and $w^2 \in M^2$,

$$\begin{aligned} & \overline{C_{M^1, M^2}}(\mathcal{Y}_{M^1, M^2}(w^1, x) w^2|_{x^n = e^{n \log 1}}) \\ &= \overline{C_{M^1, M^2}}(w^1 \boxtimes w^2) \\ &= e^{L(-1)} \overline{\mathcal{T}_{\gamma_1^-}}(w^2 \boxtimes_{P(-1)} w^1) \\ &= e^{-xL(-1)} \mathcal{Y}_{M^2, M^1}(w^2, x) w^1|_{x^n = e^{n \pi i}} \\ &= \mathcal{Y}_{M^2, M^1}^*(w^1, -x) w^2|_{x^n = e^{n \pi i}}, \end{aligned}$$

as desired. □

For any V -modules M^1, M^2, M^3 , the associativity isomorphism

$$\mathcal{A}_{M^1, M^2, M^3} : M^1 \boxtimes (M^2 \boxtimes M^3) \rightarrow (M^1 \boxtimes M^2) \boxtimes M^3,$$

was defined in [H1], [H2]. Together with the braiding isomorphism defined above, the category \mathcal{C}_V of V -module was proved to be a modular tensor category (see [H4]).

Theorem 2.13. *Let V be a vertex operator algebra satisfying conditions (1) and (2). Then the V -module category $(\mathcal{C}_V, \boxtimes)$ is a modular tensor category. For $W \in \mathcal{C}$, the dual of W is isomorphic to W' and $\theta_W = e^{2\pi i L(0)}$.*

Recall that a vertex operator algebra U is called an *extension vertex operator algebra* of V if V is a vertex operator subalgebra of U such that V, U have the same conformal vector. The following theorem was established in [HKL].

Theorem 2.14. *Let V be a vertex operator algebra satisfying conditions (1) and (2). If U is an extension vertex operator algebra of V , then U induces a commutative algebra A_U in \mathcal{C}_V such that A_U is isomorphic to U as V -module. Conversely, if U is a V -module having integral conformal weight and U is a commutative algebra in \mathcal{C}_V , then U has a vertex operator algebra structure such that U is an extension vertex operator algebra of V .*

The following result follows immediately from Theorems 2.4, 2.5, 2.13.

Theorem 2.15. *Let V be a vertex operator algebra satisfying conditions (1), (2), U be an extension vertex operator algebra of V such that U is simple and that V appears in U with multiplicity 1. Then U induces a connected etale algebra A_U in \mathcal{C}_V such that A_U is isomorphic to U as V -module.*

2.3. Tensor category of tensor product VOA. Let V_1, V_2 be vertex operator algebras satisfying conditions (1) and (2), $(\mathcal{C}_{V_1}, \boxtimes^1), (\mathcal{C}_{V_2}, \boxtimes^2)$ be the module categories of V_1, V_2 , respectively. From Theorem 2.9, $V^1 \otimes V^2$ satisfies conditions (1) and (2), then the $V^1 \otimes V^2$ -module category $(\mathcal{C}_{V^1 \otimes V^2}, \boxtimes^{12})$ is a modular tensor category. Furthermore, we have

Lemma 2.16. *For any V^1 -modules M^1, M^2 and V^2 -modules N^1, N^2 , the tensor product $(M^1 \otimes N^1) \boxtimes^{12} (M^2 \otimes N^2)$ of $M^1 \otimes N^1$ and $M^2 \otimes N^2$ is isomorphic to $(M^1 \boxtimes^1 M^2) \otimes (N^1 \boxtimes^2 N^2)$. Moreover, under this isomorphism the braiding isomorphism*

$$C_{M^1 \otimes N^1, M^2 \otimes N^2} : (M^1 \otimes N^1) \boxtimes^{12} (M^2 \otimes N^2) \rightarrow (M^2 \otimes N^2) \boxtimes^{12} (M^1 \otimes N^1)$$

is equal to $C_{M^1, M^2} \otimes C_{N^1, N^2}$.

Proof: Let $\mathcal{Y}_{M^1, M^2}, \mathcal{Y}_{N^1, N^2}$ be the intertwining operators associated to the $P(1)$ -tensor product $(M^1 \boxtimes^1 M^2, \boxtimes_{P(1)}^1), (N^1 \boxtimes^2 N^2, \boxtimes_{P(1)}^2)$, respectively. Let $\sigma(\mathcal{Y}_{M^1, M^2} \otimes \mathcal{Y}_{N^1, N^2})$ be the image of $\mathcal{Y}_{M^1, M^2} \otimes \mathcal{Y}_{N^1, N^2}$ under the isomorphism σ defined in Theorem 2.10, then $\sigma(\mathcal{Y}_{M^1, M^2} \otimes \mathcal{Y}_{N^1, N^2})$ is an intertwining operators of type

$$\left(\begin{array}{c} (M^1 \boxtimes^1 M^2) \otimes (N^1 \boxtimes^2 N^2) \\ M^1 \otimes N^1 \quad M^2 \otimes N^2 \end{array} \right)$$

From the universal property of $\mathcal{Y}_{M^1, M^2}, \mathcal{Y}_{N^1, N^2}$ and Theorems 2.9, 2.10, it is clear that $\sigma(\mathcal{Y}_{M^1, M^2} \otimes \mathcal{Y}_{N^1, N^2})$ satisfies the universal property in the definition of tensor product,

implies that $((M^1 \boxtimes^1 M^2) \otimes (N^1 \boxtimes^2 N^2), \sigma(\mathcal{Y}_{M^1, M^2} \otimes \mathcal{Y}_{N^1, N^2}))$ is a tensor product of $M^1 \otimes N^1$ and $M^2 \otimes N^2$, then there is a $V^1 \otimes V^2$ -module isomorphism

$$\phi_{M^1, M^2; N^1, N^2} : (M^1 \boxtimes^1 M^2) \otimes (N^1 \boxtimes^2 N^2) \rightarrow (M^1 \otimes N^1) \boxtimes^{12} (M^2 \otimes N^2)$$

such that

$$\mathcal{Y}_{M^1 \otimes N^1, M^2 \otimes N^2} = \phi_{M^1, M^2; N^1, N^2} \circ \sigma(\mathcal{Y}_{M^1, M^2} \otimes \mathcal{Y}_{N^1, N^2}).$$

To prove the second part of the lemma, note that the conformal vector ω of $V^1 \otimes V^2$ is equal to $\omega_1 \otimes \mathbf{1} + \mathbf{1} \otimes \omega_2$ and $L(-1) = \omega(0) = \omega_1(0) \otimes \mathbf{1} + \mathbf{1} \otimes \omega_2(0) = L_1(-1) \otimes \mathbf{1} + \mathbf{1} \otimes L_2(-1)$. Therefore, $e^{xL(-1)} = e^{x(L_1(-1) \otimes \mathbf{1})} e^{x(\mathbf{1} \otimes L_2(-1))}$. Then, by Lemma 2.12, we have for any $u^1 \in M^1, u^2 \in M^2$ and $w^1 \in N^1, w^2 \in N^2$,

$$\begin{aligned} & \overline{\phi_{M^2, M^1; N^2, N^1} \circ C_{M^1, M^2} \otimes C_{N^1, N^2} \circ \phi_{M^1, M^2; N^1, N^2}^{-1}((u^1 \otimes w^1) \boxtimes^{12} (u^2 \otimes w^2))} \\ &= \overline{\phi_{M^2, M^1; N^2, N^1} \circ C_{M^1, M^2} \otimes C_{N^1, N^2} \circ \phi_{M^1, M^2; N^1, N^2}^{-1}} \\ & \quad (\mathcal{Y}_{M^1 \otimes N^1, M^2 \otimes N^2}(u^1 \otimes w^1, x)u^2 \otimes w^2|_{x^n=e^n \log 1}) \\ &= \overline{\phi_{M^2, M^1; N^2, N^1} \circ C_{M^1, M^2} \otimes C_{N^1, N^2}} \\ & \quad (\phi_{M^1, M^2; N^1, N^2}^{-1} \circ \mathcal{Y}_{M^1 \otimes N^1, M^2 \otimes N^2}(u^1 \otimes w^1, x)u^2 \otimes w^2|_{x^n=e^n \log 1}) \\ &= \overline{\phi_{M^2, M^1; N^2, N^1} \circ C_{M^1, M^2} \otimes C_{N^1, N^2}} \\ & \quad (\sigma(\mathcal{Y}_{M^1, M^2} \otimes \mathcal{Y}_{N^1, N^2})(u^1 \otimes w^1, x)u^2 \otimes w^2|_{x^n=e^n \log 1}) \\ &= \overline{\phi_{M^2, M^1; N^2, N^1} \circ C_{M^1, M^2} \otimes C_{N^1, N^2}} \\ & \quad ((\mathcal{Y}_{M^1, M^2}(u^1, x)u^2) \otimes (\mathcal{Y}_{N^1, N^2}(w^1, x)w^2)|_{x^n=e^n \log 1}) \\ &= \overline{\phi_{M^2, M^1; N^2, N^1}}((\mathcal{Y}_{M^2, M^1}^*(u^1, -x)u^2) \otimes (\mathcal{Y}_{N^2, N^1}^*(w^1, -x)w^2)|_{x^n=e^n \pi i}) \\ &= \overline{\phi_{M^2, M^1; N^2, N^1}}(\sigma(\mathcal{Y}_{M^2, M^1}^* \otimes \mathcal{Y}_{N^2, N^1}^*)(u^1 \otimes w^1, -x)u^2 \otimes w^2|_{x^n=e^n \pi i}) \\ &= \mathcal{Y}_{M^2 \otimes N^2, M^1 \otimes N^1}^*(u^1 \otimes w^1, -x)u^2 \otimes w^2|_{x^n=e^n \pi i} \\ &= \overline{C_{M^1 \otimes N^1, M^2 \otimes N^2}}((u^1 \otimes w^1) \boxtimes^{12} (u^2 \otimes w^2)). \end{aligned}$$

Thus

$$C_{M^1 \otimes N^1, M^2 \otimes N^2} = \phi_{M^2, M^1; N^2, N^1} \circ C_{M^1, M^2} \otimes C_{N^1, N^2} \circ \phi_{M^1, M^2; N^1, N^2}^{-1}.$$

This finishes the proof. \square

Lemma 2.17. *For any V^1 -module M and V^2 -module N , the dual $(M \otimes N)'$ of $M \otimes N$ is isomorphic to $M' \otimes N'$.*

Proof: This follows immediately from Lemma 2.16 and the uniqueness of the dual of $M \otimes N$. \square

Lemma 2.18. *For any V^1 -module M and V^2 -module N , we have*

$$\text{FPdim}_{c_{V^1 \otimes V^2}} M \otimes N = \text{FPdim}_{c_{V^1}} M \cdot \text{FPdim}_{c_{V^2}} N.$$

Proof: For any V^1 -module M and V^2 -module N , define $\phi(M \otimes N) = \text{FPdim}_{\mathcal{C}_{V^1}} M \cdot \text{FPdim}_{\mathcal{C}_{V^2}} N$. From Lemma 2.16, ϕ is a homomorphism from the Grothendieck ring of $\mathcal{C}_{V^1 \otimes V^2}$ to real numbers sending each isomorphism class to a nonnegative real number. Since homomorphism having this property is unique, we have $\text{FPdim}_{\mathcal{C}_{V^1 \otimes V^2}} M \otimes N = \text{FPdim}_{\mathcal{C}_{V^1}} M \cdot \text{FPdim}_{\mathcal{C}_{V^2}} N$. \square

3. MIRROR EXTENSION OF VERTEX OPERATOR ALGEBRAS

In this section, we shall prove our main results. Let V^1, V^2 be vertex operator algebras satisfying conditions (1) and (2). Denote the module categories of V^1, V^2 by $\mathcal{C}_{V^1}, \mathcal{C}_{V^2}$, respectively. If V is an extension vertex operator algebra of $V^1 \otimes V^2$, it is known that $V^1 \otimes V^2$ is rational by Theorem 2.9, then V has the following decomposition as $V^1 \otimes V^2$ -module:

$$V = \bigoplus_{i \in I, j \in J} Z_{i,j} M^i \otimes N^j,$$

where $Z_{i,j}$, ($i \in I, j \in J$), are nonnegative integers and $\{M^i | i \in I\}$ (resp. $\{N^j | j \in J\}$) are inequivalent irreducible V^1 -modules (resp. V^2 -modules) such that $Z_{i,j} \neq 0$ for some $(i, j) \in I \times J$. In the following we assume that V is an extension vertex operator algebra of $V^1 \otimes V^2$ such that V is simple, self dual and $\text{Hom}_{V^1 \otimes V^2}(V^1 \otimes N^j, V) = \mathbb{C}$ (resp. $\text{Hom}_{V^1 \otimes V^2}(M^i \otimes V^2, V) = \mathbb{C}$) if and only if $N^j = V^2$ (resp. $M^i = V^1$). Then we have

Theorem 3.1. (i) $Z_{i,j} = 1$ if $Z_{i,j} \neq 0$. Moreover, this induces a bijection τ from I to J such that $\tau(i) = j$ if and only if $Z_{i,j} = 1$.

(ii) The set $\{M^i | i \in I\}$ (resp. $\{N^j | j \in J\}$) is closed under the tensor product of category \mathcal{C}_{V^1} (resp. \mathcal{C}_{V^2}).

(iii) For any $i_1, i_2, i_3 \in I$, $N_{M^{i_1}, M^{i_2}}^{M^{i_3}} = N_{(N^{\tau(i_1)})', (N^{\tau(i_2)})'}^{(N^{\tau(i_3)})'}$.

Proof: Let $\{M^i | i \in I^c\}$ (resp. $\{N^j | j \in J^c\}$) be irreducible V^1 -modules (resp. V^2 -modules) such that $\{M^i | i \in I\}$ (resp. $\{N^j | j \in J\}$) is a subset of $\{M^i | i \in I^c\}$ (resp. $\{N^j | j \in J^c\}$) and is closed under the tensor product of category \mathcal{C}_{V^1} (resp. \mathcal{C}_{V^2}). For $i \in I^c, j \in J^c$, we take $Z_{i,j} = 0$ if $i \in I^c \setminus I$ or $j \in J^c \setminus J$, then we have $V = \bigoplus_{i \in I^c, j \in J^c} Z_{i,j} M^i \otimes N^j$.

Consider the module category $\mathcal{C}_{V^1 \otimes V^2}$ of $V^1 \otimes V^2$, from Theorem 2.15, V induces an etale algebra A in $\mathcal{C}_{V^1 \otimes V^2}$ such that A is isomorphic to V as $V^1 \otimes V^2$ -module. From Proposition 2.1, Theorem 2.6 and Lemmas 2.16, 2.17, we have for any $i_1, i_2 \in I^c$,

$$\begin{aligned} & \text{Hom}_A(A \boxtimes^{12} (M^{i_1} \otimes V^2), A \boxtimes^{12} (M^{i_2} \otimes V^2)) \\ &= \text{Hom}(M^{i_1} \otimes V^2, A \boxtimes^{12} (M^{i_2} \otimes V^2)) \\ &= \text{Hom}((M^{i_1} \otimes V^2) \boxtimes^{12} (M^{i_2} \otimes V^2)', A) \\ &= \text{Hom}((M^{i_1} \boxtimes^1 (M^{i_2})') \otimes V^2, A), \end{aligned}$$

it follows from the assumption that $\text{Hom}_A(A \boxtimes^{12} (M^{i_1} \otimes V^2), A \boxtimes^{12} (M^{i_2} \otimes V^2)) = \mathbb{C}$ if and only if $M^{i_1} = M^{i_2}$. This implies that $\{A \boxtimes^{12} (M^i \otimes V^2) | i \in I^c\}$ are simple left A -modules and mutually inequivalent. Similarly, we have $\text{Hom}_A(A \boxtimes^{12} (V^1 \otimes N^{j_1}), A \boxtimes^{12} (V^1 \otimes N^{j_2})) = \mathbb{C}$ if and only if $N^{j_1} = N^{j_2}$. Therefore, $\{A \boxtimes^{12} (V^1 \otimes N^j) | j \in J^c\}$ are simple left A -modules and mutually inequivalent.

Also from Proposition 2.1, Theorem 2.6 and Lemmas 2.16, 2.17, we have for any $i \in I^c$ and $j \in J^c$,

$$\begin{aligned} & \text{Hom}_A(A \boxtimes^{12} (M^i \otimes V^2), A \boxtimes^{12} (V^1 \otimes N^j)) \\ &= \text{Hom}((M^i \otimes V^2), A \boxtimes^{12} (V^1 \otimes N^j)) \\ &= \text{Hom}((M^i \otimes V^2) \boxtimes^{12} (V^1 \otimes N^j)', A) \\ &= \text{Hom}(M^i \otimes (N^j)', A), \end{aligned}$$

implies that $\dim \text{Hom}_A(A \boxtimes^{12} (M^i \otimes V^2), A \boxtimes^{12} (V^1 \otimes (N^j)')) = Z_{i,j}$ and that $Z_{i,j} = 1$ if $Z_{i,j} \neq 0$. Similarly, we have $\dim \text{Hom}_A(A \boxtimes^{12} (V^1 \otimes (N^j)'), A \boxtimes^{12} (M^i \otimes V^2)) = Z_{i',j'}$ and $Z_{i',j'} = 1$ if $Z_{i',j'} \neq 0$, where we use $N^{j'}$ to denote $(N^j)'$. Note that V is self dual, therefore $Z_{i,j} = Z_{i',j'}$. This implies that $A \boxtimes^{12} (M^i \otimes V^2)$ is isomorphic to $A \boxtimes^{12} (V^1 \otimes (N^j)')$ as left A -module if $Z_{i,j} \neq 0$ and that $A \boxtimes^{12} (M^i \otimes V^2)$ is not isomorphic to any one of $\{A \boxtimes^{12} (V^1 \otimes N^j) | j \in J^c\}$ if $i \in I^c \setminus I$. Since $\{A \boxtimes^{12} (M^i \otimes V^2) | i \in I^c\}$ (resp. $\{A \boxtimes^{12} (V^1 \otimes N^j) | j \in J^c\}$) are inequivalent left A -modules, for any $i \in I$ there is a unique $j \in J$ such that $Z_{i,j} \neq 0$. Therefore, we have a bijection τ from I to J such that $\tau(i) = j$ if and only if $Z_{i,j} \neq 0$.

For any $i \in I, j \in J$, consider the left A -module $A \boxtimes^{12} (M^i \otimes (N^j)'),$ from the discussion above and Lemmas 2.16, 2.17, we have

$$\begin{aligned} A \boxtimes^{12} (M^i \otimes (N^j)') &\cong A \boxtimes^{12} ((M^i \otimes V^2) \boxtimes^{12} (V^1 \otimes (N^j)')) \\ &\cong (A \boxtimes^{12} (M^i \otimes V^2)) \boxtimes^{12} (V^1 \otimes (N^j)') \\ &\cong (A \boxtimes^{12} (V^1 \otimes (N^{\tau(i)}))) \boxtimes^{12} (V^1 \otimes (N^j)') \\ &\cong A \boxtimes^{12} ((V^1 \otimes (N^{\tau(i)}))' \boxtimes^{12} (V^1 \otimes (N^j)')) \\ &\cong A \boxtimes^{12} (V^1 \otimes ((N^{\tau(i)})' \boxtimes^2 (N^j)')) \\ &\cong \oplus_l N_{(N^{\tau(i)})', (N^j)'}^{N^l} A \boxtimes^{12} (V^1 \otimes N^l). \end{aligned}$$

On the other hand,

$$\begin{aligned}
A \boxtimes^{12} (M^i \otimes (N^j)') &\cong A \boxtimes^{12} ((V^1 \otimes (N^j)') \boxtimes^{12} (M^i \otimes V^2)) \\
&\cong (A \boxtimes^{12} (V^1 \otimes (N^j)')) \boxtimes^{12} (M^i \otimes V^2) \\
&\cong (A \boxtimes^{12} (M^{\tau^{-1}(j)} \otimes V^2)) \boxtimes^{12} (M^i \otimes V^2) \\
&\cong A \boxtimes^{12} ((M^{\tau^{-1}(j)} \otimes V^2) \boxtimes^{12} (M^i \otimes V^2)) \\
&\cong A \boxtimes^{12} ((M^{\tau^{-1}(j)} \boxtimes^1 M^i) \otimes V^2) \\
&\cong \oplus_k N_{M^{\tau^{-1}(j)}, M^i}^{M^k} A \boxtimes^{12} (M^k \otimes V^2).
\end{aligned}$$

Comparing the two decompositions, we obtain that the set $\{M^i | i \in I\}$ (resp. $\{N^j | j \in J\}$) is closed under the tensor product of category \mathcal{C}_{V_1} (resp. \mathcal{C}_{V_2}) and that $N_{M^{\tau^{-1}(i')}, M^i}^{M^{\tau^{-1}(i')}} = N_{(N^j)', (N^{\tau(i)})'}^{N^i}$. Note that $\tau(i') = \tau(i)'$, then we have $N_{M^{i_1}, M^{i_2}}^{M^{i_3}} = N_{(N^{\tau(i_1)})', (N^{\tau(i_2)})'}^{(N^{\tau(i_3)})'}$. The proof is complete. \square

From Theorem 3.1, we have $\{M^i | i \in I\}$ (resp. $\{N^j | j \in J\}$) is closed under the tensor product of \mathcal{C}_{V_1} (resp. \mathcal{C}_{V_2}). Let $\mathcal{C}_{V_1}^0$ (resp. $\mathcal{C}_{V_2}^0$) be the fusion subcategory of \mathcal{C}_{V_1} (resp. \mathcal{C}_{V_2}) such that the Grothendieck ring of $\mathcal{C}_{V_1}^0$ (resp. $\mathcal{C}_{V_2}^0$) is equal to $\{M^i | i \in I\}$ (resp. $\{N^j | j \in J\}$). $\mathcal{C}_{V_1}^0$ (resp. $\mathcal{C}_{V_2}^0$) can be viewed as the subcategory of $\mathcal{C}_{V_1 \otimes V_2}$ by identifying M^i (resp. N^j) with $M^i \otimes V^2$ (resp. $V^1 \otimes N^j$). Note that $\{M^i \otimes N^j | i \in I, j \in J\}$ is closed under the tensor product of $\mathcal{C}_{V_1 \otimes V_2}$. Let \mathcal{C} be the fusion subcategory of $\mathcal{C}_{V_1 \otimes V_2}$ such that the Grothendieck ring of \mathcal{C} is equal to $\{M^i \otimes N^j | i \in I, j \in J\}$. Therefore, A is a connected etale algebra in \mathcal{C} . Let \mathcal{C}_A be the category of A -module in \mathcal{C} . Then we have

Lemma 3.2. $\text{FPdim } \mathcal{C}_A = \text{FPdim } \mathcal{C}_{V_1}^0 = \text{FPdim } \mathcal{C}_{V_2}^0$.

Proof: From Theorem 3.1, we get for any $i \in I$, $\text{FPdim}_{\mathcal{C}_{V_1}^0} M^i = \text{FPdim}_{\mathcal{C}_{V_2}^0} (N^{\tau(i)})' = \text{FPdim}_{\mathcal{C}_{V_2}^0} N^{\tau(i)}$, thus $\text{FPdim } \mathcal{C}_{V_1}^0 = \text{FPdim } \mathcal{C}_{V_2}^0$. It follows from Lemma 2.18 that $\text{FPdim}_{\mathcal{C}} A = \text{FPdim } \mathcal{C}_{V_1}^0 = \text{FPdim } \mathcal{C}_{V_2}^0$. Note that $\text{FPdim } \mathcal{C} = \text{FPdim } \mathcal{C}_{V_1}^0 \cdot \text{FPdim } \mathcal{C}_{V_2}^0$, from Proposition 2.7 we have $\text{FPdim } \mathcal{C}_A = \text{FPdim } \mathcal{C}_{V_1}^0 = \text{FPdim } \mathcal{C}_{V_2}^0$. \square

The following result plays an important role in the proof of our main results.

Theorem 3.3. *There is a braid-reversing equivalence*

$$\mathcal{T} : \mathcal{C}_{V_1}^0 \rightarrow \mathcal{C}_{V_2}^0$$

such that \mathcal{T} sends $M^i \in \mathcal{C}_{V_1}^0$ to $(N^{\tau(i)})' \in \mathcal{C}_{V_2}^0$.

Proof: The proof is similar to that of Theorem 5.1 in [OS]. Let \mathcal{C} and \mathcal{C}_A the categories defined above. Consider the free module functor

$$\mathcal{F}_A : \mathcal{C} \rightarrow \mathcal{C}_A, \quad X \mapsto X \boxtimes^{12} A.$$

We now prove that the restriction of \mathcal{F}_A to the subcategory $\mathcal{C}_{V^1}^0$ of \mathcal{C} is fully faithful, that is $\text{Hom}_{\mathcal{C}_{V^1}^0}(M^1, M^2) = \text{Hom}_{\mathcal{C}_A}(\mathcal{F}_A(M^1), \mathcal{F}_A(M^2))$ for any $M^1, M^2 \in \mathcal{C}_{V^1}^0$. In fact, from Proposition 2.1, Theorem 2.6 and Lemmas 2.16, 2.17, we have

$$\begin{aligned}
\text{Hom}_{\mathcal{C}_A}(\mathcal{F}_A(M^1), \mathcal{F}_A(M^2)) &= \text{Hom}_{\mathcal{C}_A}((M^1 \otimes V^2) \boxtimes^{12} A, (M^2 \otimes V^2) \boxtimes^{12} A) \\
&= \text{Hom}(M^1 \otimes V^2, (M^2 \otimes V^2) \boxtimes^{12} A) \\
&= \text{Hom}((M^2 \otimes V^2)' \boxtimes^{12} (M^1 \otimes V^2), A) \\
&= \text{Hom}(((M^2)') \boxtimes^1 M^1) \otimes V^2, A) \\
&= \text{Hom}(((M^2)') \boxtimes^1 M^1) \otimes V^2, V^1 \otimes V^2) \\
&= \text{Hom}(M^1 \otimes V^2, M^2 \otimes V^2) \\
&= \text{Hom}(M^1, M^2).
\end{aligned}$$

Similarly, the restriction of \mathcal{F}_A to the subcategory $\mathcal{C}_{V^2}^0$ is fully faithful. It follows from Lemma 3.2 that \mathcal{F}_A restricted to $\mathcal{C}_{V^2}^0$ and $\mathcal{C}_{V^2}^0$ are monoidal equivalences, and then $\mathcal{C}_{V^1}^0$ and $\mathcal{C}_{V^2}^0$ are monoidal equivalent.

Recall that the free module functor \mathcal{F}_A has the structure of a central functor given by $X \mapsto (X \boxtimes^{12} A, C_{-,X})$, where $C_{-,X}$ denotes the braiding isomorphism in \mathcal{C} , then we have two fully faithful braided monoidal functors $\mathcal{F}_1 : \mathcal{C}_{V^1}^0 \rightarrow \mathcal{Z}(\mathcal{C}_A)$ and $\mathcal{F}_2 : \mathcal{C}_{V^2}^0 \rightarrow \mathcal{Z}(\mathcal{C}_A)$. Furthermore, from Theorem 2.2 and Lemma 2.16 we have

$$\mathcal{F}_1(\mathcal{C}_{V^1}^0) \subset (\mathcal{F}_2(\mathcal{C}_{V^2}^0))',$$

where $(\mathcal{F}_2(\mathcal{C}_{V^2}^0))'$ denotes the centralizer of $\mathcal{F}_2(\mathcal{C}_{V^2}^0)$.

On the other hand, since \mathcal{C}_A is monoidal equivalent to $\mathcal{C}_{V^2}^0$, we could consider \mathcal{C}_A as a braided monoidal category with the braiding induced from $\mathcal{C}_{V^2}^0$. It follows from Theorem 2.3 that there exist fully faithful braided monoidal functor $i : \mathcal{C}_A \rightarrow \mathcal{Z}(\mathcal{C}_A)$ and braid-reversing monoidal functor $j : \mathcal{C}_A \rightarrow \mathcal{Z}(\mathcal{C}_A)$ such that $(i(\mathcal{C}_A))' = j(\mathcal{C}_A)$. Note that the image of i is the same as the image of \mathcal{F}_2 , hence $(\mathcal{F}_2(\mathcal{C}_{V^2}^0))' = j(\mathcal{C}_A)$. Since j and \mathcal{F}_1 are fully faithful, we have $\text{FPdim}(\mathcal{F}_2(\mathcal{C}_{V^2}^0))' = \text{FPdim } j(\mathcal{C}_A) = \text{FPdim } \mathcal{C}_A = \text{FPdim } \mathcal{C}_{V^1}^0 = \text{FPdim } \mathcal{F}_1(\mathcal{C}_{V^1}^0)$, implies $\mathcal{F}_1(\mathcal{C}_{V^1}^0) = (\mathcal{F}_2(\mathcal{C}_{V^2}^0))' = j(\mathcal{C}_A)$. Let \mathcal{T} be the following composition functor

$$\mathcal{T} : \mathcal{C}_{V^1}^0 \xrightarrow{\mathcal{F}_1} \mathcal{F}_1(\mathcal{C}_{V^1}^0) \xrightarrow{j^{-1}} \mathcal{C}_A \xrightarrow{i} \mathcal{F}_2(\mathcal{C}_{V^2}^0) \xrightarrow{\mathcal{F}_2^{-1}} \mathcal{C}_{V^2}^0,$$

where all functors present except for j^{-1} are braided with j^{-1} are braid-reversing. Then we have $\mathcal{T} : \mathcal{C}_{V^1}^0 \rightarrow \mathcal{C}_{V^2}^0$ is a braid-reversing equivalence.

From Proposition 2.1, Theorem 2.6 and Lemmas 2.16, 2.17, we have

$$\begin{aligned}
\text{Hom}_{\mathcal{C}_A}((M^i \otimes V^2) \boxtimes^{12} A, (V^1 \otimes N^j) \boxtimes^{12} A) &= \text{Hom}(M^i \otimes V^2, (V^1 \otimes N^j) \boxtimes^{12} A) \\
&= \text{Hom}((V^1 \otimes N^j)' \boxtimes^{12} (M^i \otimes V^2), A) \\
&= \text{Hom}(M^i \otimes (N^j)', A).
\end{aligned}$$

It follows that $\text{Hom}_{\mathcal{C}_A}((M^i \otimes V^2) \boxtimes^{12} A, (V^1 \otimes N^j) \boxtimes^{12} A) = \mathbb{C}$ if and only if $j = \tau(i)'$. Therefore, $\mathcal{T}(M^i) = (N^{\tau(i)})'$. The proof is complete. \square

Remark 3.4. When V^1 is the affine vertex operator algebra $L_{sl_m}(n, 0)$ and V^2 is the affine vertex operator algebra $L_{sl_n}(m, 0)$, the result in Theorem 3.3 has been obtained in [OS], and gives an explanation of the level-rank duality [F]. From this point of view, the vertex operator algebra V^1 is dual to V^2 .

We are now in a position to prove our main result in this paper.

Theorem 3.5. Let $M^0 = V^1, M^1, \dots, M^k$ be simple objects in $\mathcal{C}_{V^1}^0$ such that

$$U = M^0 \oplus M^1 \oplus \dots \oplus M^k$$

is an extension vertex operator algebra of V^1 . Then

$$U^c = V^2 \oplus (N^{\tau(1)})' \oplus \dots \oplus (N^{\tau(k)})'$$

has a vertex operator algebra structure such that U^c is an extension vertex operator algebra of V^2 . Moreover, U^c is a simple vertex operator algebra if U is a simple vertex operator algebra.

Proof: It follows immediately from Theorems 2.14, 3.3 that U^c induces an algebra (A_{U^c}, μ) in the category $\mathcal{C}_{V^2}^0$ such that

$$\mu \circ C_{U^c, U^c}^{-1} = \mu.$$

Note that the conformal weight of U^c is integral, by e.q. (2.8) in [HKL], we have

$$C_{U^c, U^c}^{-1} = C_{U^c, U^c} \circ \theta_{U^c \boxtimes^2 U^c},$$

implies

$$\begin{aligned} \mu &= \mu \circ C_{U^c, U^c}^{-1} \\ &= \mu \circ C_{U^c, U^c} \circ \theta_{U^c \boxtimes^2 U^c} \\ &= \mu \circ C_{U^c, U^c}, \end{aligned}$$

where in the last equality we have used the facts that μ is a V^2 -module morphism and that the conformal weight of U^c is integral. Thus (A_{U^c}, μ) is a commutative algebra in the category $\mathcal{C}_{V^2}^0$. Then it follows from Theorem 2.14 that there is a vertex operator algebra structure on U^c .

Assume that U is a simple vertex operator algebra. We now prove that U^c is a simple vertex operator algebra. Otherwise, U^c has a nontrivial ideal U^1 , then it follows from Theorem 3.4 in [HKL] that U^1 induces a nontrivial A_{U^c} -submodule (U^1, μ_{U^1}) of A_{U^c} such that

$$\mu_{U^1} \circ C_{U^1, U^c} \circ C_{U^c, U^1} = \mu_{U^1}.$$

Thus, by Theorem 3.3, A_U has a nontrivial A_U -submodule $(\tilde{U}^1, \mu_{\tilde{U}^1})$ such that

$$\mu_{\tilde{U}^1} \circ C_{\tilde{U}^1, U}^{-1} \circ C_{U, \tilde{U}^1}^{-1} = \mu_{\tilde{U}^1},$$

where A_U is the etale algebra induced from the vertex operator algebra U . Since the conformal weight of U is integral, by a similar argument as above, we have

$$\mu_{\tilde{U}^1} \circ C_{\tilde{U}^1, U} \circ C_{U, \tilde{U}^1} = \mu_{\tilde{U}^1}.$$

It follows from Theorem 3.4 in [HKL] that the vertex operator algebra U has a nontrivial ideal, this is a contradiction. The proof is complete. \square

As an application, we prove the mirror extension conjecture in [DJX1]. First, recall that if $(U, Y, \mathbf{1}, \omega)$ is a vertex operator algebra and $(V, Y, \mathbf{1}, \omega')$ is a vertex operator subalgebra of U such that $\omega' \in U_2$ and $L(1)\omega' = 0$, then $(V^c, Y, \mathbf{1}, \omega - \omega')$ is a vertex operator subalgebra of U , where $V^c = \{v \in U | L'(-1)v = 0\}$ (see [FZ]). In the following we assume that $(V, Y, \mathbf{1}, \omega)$, $(V^c, Y, \mathbf{1}, \omega - \omega')$, $(U, Y, \mathbf{1}, \omega')$ satisfy conditions (1), (2) and $(V^c)^c = V$, it follows from Theorem 3.1 that U has the decomposition

$$U = V \otimes V^c \oplus (\oplus_{i=1}^n M^i \otimes N^i)$$

such that $\{M^i | 1 \leq i \leq n\}$ (resp. $\{N^i | 1 \leq i \leq n\}$) are mutually inequivalent irreducible V -modules (resp. V^c -modules). Let

$$V^e = V \oplus (\oplus_{i=1}^n m_i M^i)$$

and

$$(V^c)^e = V^c \oplus (\oplus_{i=1}^n m_i (N^i)'),$$

where m_i , $1 \leq i \leq n$, are nonnegative integers. Then we have

Theorem 3.6. *If there is a vertex operator algebra structure on V^e such that V^e is an extension vertex operator algebra of V . Then $(V^c)^e$ has a vertex operator algebra structure such that $(V^c)^e$ is an extension vertex operator algebra of V^c . Moreover, $(V^c)^e$ is a simple vertex operator algebra if V^e is a simple vertex operator algebra.*

Proof: This follows immediately from Theorem 3.5. \square

REFERENCES

- [ABD] T. Abe, G. Buhl and C. Dong, Rationality, regularity and C_2 -cofiniteness, *Trans. Amer. Math. Soc.* **356** (2004), 3391-3402.
- [ADL] T. Abe, C. Dong and H. Li, Fusion rules for the vertex operator algebras $M(1)^+$ and V_L^+ , *Comm. Math. Phys.* **253** (2005), 171-219.
- [BK] B. Bakalov and A. Kirillov, Lectures on tensor categories and modular functors. *University Lecture Series* Vol 21, 2000.
- [DMNO] A. Davydov, M. Muger, D. Nikshych and V. Ostrik, The Witt group of non-degenerate braided fusion categories, *J. Reine Angew. Math.* **677** (2013), 135-177.

- [DJX1] C. Dong, X. Jiao and F. Xu, Mirror extensions of vertex operator algebras, *Comm. Math. Phys.* **329** (2014), 263-294.
- [DJX2] C. Dong, X. Jiao and F. Xu, Quantum dimensions and quantum Galois theory, *Trans. Amer. Math. Soc.* **365** (2013), 6441-6469.
- [DLM] C. Dong, H. Li and G. Mason, Regularity of rational vertex operator algebras, *Adv. Math.* **132** (1997), 148-166.
- [DMZ] C. Dong, G. Mason and Y. Zhu, Discrete series of the Virasoro algebra and the moonshine module, *Proc. Symp. Pure. Math. AMS* **56** II (1994), 295-316.
- [ENO] P. Etingof, D. Nikshych and V. Ostrik, On fusion categories, *Ann. of Math.* **162** (2005), 581-642.
- [F] I. Frenkel, Representations of affine Lie algebras, Hecke modular forms and Korteweg de Vries type equations. *Lecture Notes in Math.* **933**. Berlin-Heidelberg-New York: Springer, 1981. 71-109
- [FHL] I. Frenkel, Y. Huang and J. Lepowsky, On axiomatic approaches to vertex operator algebras and modules, *Mem. Amer. Math. Soc.* **104**, 1993.
- [FLM] I. Frenkel, J. Lepowsky and A. Meurman, Vertex operator algebras and the Monster, *Pure and Appl. Math.* Vol 134, 1988.
- [FZ] I. Frenkel and Y. Zhu, Vertex operator algebras associated to representations of affine and Virasoro algebras, *Duke Math. J.* **66** (1992), 123-168.
- [H1] Y. Huang, A theory of tensor product for module categories for a vertex operator algebra, IV, *J. Pure Appl. Algebra* **100** (1995), 173-216.
- [H2] Y. Huang, Virasoro vertex operator algebra, the (nonmeromorphic) operator product expansion and the tensor product theory, *J. Algebra* **182** (1996), 201-234.
- [H3] Y. Huang, Differential equations and intertwining operators, *Comm. Contemp. Math.* **7** (2005), 375-400.
- [H4] Y. Huang, Rigidity and modularity of vertex operator algebras, *Comm. Contemp. Math.* **10** (2008), 871-911.
- [HKL] Y. Huang, A. Kirillov and J. Lepowsky, Braided tensor categories and extensions of vertex operator algebras, *arXiv 1406.3420*.
- [HL1] Y. Huang and J. Lepowsky, A theory of tensor products for module categories for a vertex operator algebra, I, *Selecta. Math. (N. S.)* **1** (1995), 699-756.
- [HL2] Y. Huang and J. Lepowsky, A theory of tensor products for module categories for a vertex operator algebra, II, *Selecta. Math. (N. S.)* **1** (1995), 756-786.
- [HL3] Y. Huang and J. Lepowsky, A theory of tensor products for module categories for a vertex operator algebra, III, *J. Pure Appl. Algebra* **100** (1995), 141-171.
- [HL4] Y. Huang and J. Lepowsky, Tensor products of modules for a vertex operator algebra and vertex tensor categories, in: *Lie Theory and Geometry, in honor of Bertram Kostant*, ed. R. Brylinski, J.-L. Brylinski, V. Guillemin, V. Kac, Birkhäuser, Boston, 1994, 349-383.
- [Ka] C. Kassel, Quantum groups, Graduate Texts in Mathematics, Berlin-Heidelberg-New York: Springer-Verlag, 1995.
- [KO] A. Kirillov and V. Ostrik, On a q-analog of McKay correspondence and ADE classification of \widehat{sl}_2 conformal field theories, *Adv. Math.* **171**(2002), 183-227.
- [LL] J. Lepowsky and H. Li, Introduction to vertex operator algebras and their representations, *Progress in Math.* Vol 227, 2004.
- [L] H. Li, Some finiteness properties of regular vertex operator algebras, *J. Algebra* **212** (1999), 495-514.
- [M1] M. Muger, On the structure of modular categories, *Proc. Lond. Math. Soc.* **87** (2003), 291-308.
- [M2] M. Muger, From subfactors to categories and topology. II. The quantum double of tensor categories and subfactors, *J. Pure Appl. Algebra* **180** (2003), 159-219.

- [NT] T. Nakanishi and A. Tsuchiya, Level-rank duality of WZW models in conformal field theory, *Commun. Math. Phys.* **144** (1992), 351-372.
- [O] V. Ostrik, Module categories, weak hopf algebras and modular invariants, *Transform. Groups* **8** (2003), 177-206.
- [OS] V. Ostrik and M. Sun, Level-rank dual via tensor category, *Commun. Math. Phys.* **326** (2014), 49-61.
- [X] F. Xu, Mirror extensions of local nets, *Commun. Math. Phys.* **270** (2007), 835-847.
- [Z] Y. Zhu, Modular invariance of characters of vertex operator algebras, *J. Amer. Math. Soc.* **9** (1996), 237-302.

XINGJUN LIN, INSTITUTE OF MATHEMATICS, ACADEMIA SINICA, TAIPEI 10617, TAIWAN
E-mail address: `linxingjun@math.sinica.edu.tw`